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DYNAMIC RESPONSE VARIABILITY OF STRUCTURES WITH UNCERTAIN PROPERTIES

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SUMMARY

A modal-based analysis of the dynamic response variability of multiple degree-of-freedom linear structures with uncertain parameters subjected to either deterministic or stochastic excitations is considered. A probabilistic methodology is presented in which random variables with specified probability distributions are used to quantify the parameter uncertainties. The uncertainty in the response due to uncertainties in the structural modelling and loading is quantified by various probabilistic measures such as mean, variance and coefficient of excess. The computation of these probabilistic measures is addressed. A series expansion involving orthogonal polynomials in terms of the system parameters is first used to model the response variability of each contributing mode. Linear equations for the coefficients of each series expansion are derived using the weighted residual method. Mode superposition is then used to derive analytical expressions for the variability and statistics of the uncertain response in terms of the coefficients of the series expansions for all contributing modes. A primary–secondary system and a ten-story building subjected to deterministic and stochastic loads are used to demonstrate the methodology, as well as evaluate its performance by comparing it to existing methods, including the computationally cost-efficient perturbation method.

KEY WORDS: uncertain dynamics; random vibration; MDOF structures; polynomial expansions; response statistics

INTRODUCTION

Uncertainties in both loading and structural characteristics can adversely affect the response and reliability of a structure. Therefore, it is often desirable to consider the effects of these uncertainties in the analysis and design. This requires the development of appropriate mathematical tools for incorporating such uncertainties in the structural modelling, as well as the development of methods for analyzing the resulting mathematical model. Random processes, for example, are often used as mathematical tools for modelling the uncertain load time histories, such as those due to earthquakes or winds. Random vibration theory can be used to obtain useful probabilistic descriptors of the response, such as mean and covariance for Gaussian processes, as well as higher-order statistics for non-Gaussian processes. These probabilistic descriptors can be combined with approximate extreme response theories to obtain various reliability estimates for the structure.

An increasing interest has been shown in treating the uncertainties involved in the modelling of the structural characteristics. These uncertainties, which result from the numerous assumptions made when modelling the geometry, the boundary conditions and the constitutive behavior of the materials involved, can have a significant effect on the reliability of several structures.[1–4] Therefore, when considering a structural model, such as a finite element model, one must account for uncertainties[5,6] in element or substructure stiffness, mass, modal damping coefficients, spring constants modelling the boundary conditions, or equivalent soil stiffness when accounting for soil–structure interaction.

Probabilistic methods provide the means of incorporating structural modelling uncertainties in the analysis of the system response by describing the uncertainties as random variables. Thus, random variables are used as appropriate mathematical tools for modelling uncertainties in discrete parameters in the same way as, for example, random processes are used to model the uncertain temporal variation of earthquake or wind loads. The probability distribution assigned to each random variable describes how plausible each possible value is for the corresponding model parameter. Uncertainties in continuous variables, such as the spatial variation of

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the modulus of elasticity of a structural member, can also be quantified by random fields possessing certain correlation structure. Research in this area has focused on modelling and incorporating the spatial distribution of the uncertainties into a finite element methodology. For this purpose, different methods of discretization of the random fields have been employed in the past, leading to various probabilistic finite element methodologies[7–13] in which uncertainties are described by a discrete set of random variables.

In the present work an analytical method based on modal analysis is developed for computing various statistical measures of the uncertain response due to uncertainties in both the structural and loading characteristics. The type of loads considered are both deterministic and stochastic. Existing methods, based on simulations or numerical integration over the parameter space, are powerful but they are very costly and time consuming because they require a large number of repeated dynamic analyses of the structure. Perturbation techniques and second-moment analysis based on Taylor series expansions,[10,14] although they are computationally the least expensive methods, only work well for limited cases such as static or eigenvalue problems with small uncertainties. For dynamic problems of the type addressed in this study, they suffer from inaccuracy even if the level of uncertainties are small.[6,15–17] Moreover, as the order of the Taylor expansion increases, the accuracy of the results may deteriorate and convergence is not guaranteed.[15]

Approaches based on an expansion of the response in a series of orthogonal functions of the uncertain system parameters overcome these deficiencies, have better convergence properties and provide accurate results,[6,9,18] but usually require more computational effort than perturbation methods. Such orthogonal expansion methods (OEM) have been used to solve both static and dynamic problems. Also, they have been extended to calculate the variability of the response moments of systems subjected to stochastic excitations due to variability of the system parameters.[1] However, for large or even medium-sized systems, and for cases where many uncertain variables are involved, the computational cost and computer storage requirements of OEM methods may become excessive. Furthermore, previous OEM methods have limitations associated with the complexity of the uncertain structural properties and the type of probability distributions they can handle.

In this study we propose an OEM method based on modal analysis of linear systems. Both the cases of deterministic and stochastic input are considered. The general case of non-classical modes is treated. The method becomes computationally more efficient than previous OEM methods by considering only the contributing modes. Analytical expressions are derived for the response variability as a function of the uncertain system parameters. Using this variability and the uncertainties associated with the system parameters, the uncertainty of the response can be quantified. Probabilistic quantities such as mean, variance, and the coefficient of excess which is useful in reliability studies, are used to quantify the uncertain response. Analytical expressions are derived for computing these measures. The advantages of the present method in terms of required computational time, computer storage, as well as accuracy are discussed. Selected numerical case studies are used to test the proposed methodology and investigate the effect of the system uncertainties on the response. Comparisons with methods that converge to the exact solutions, such as numerical integration or simulations, show that the proposed method yields very good results while the computationally least expensive perturbation method may produce very inaccurate results even when the level of uncertainties considered is relatively small.

SUMMARY OF MODAL-BASED FORMULATION IN STRUCTURAL DYNAMICS

Deterministic excitation

The equation of motion of an n degree-of-freedom linear structure subjected to external forces can be cast in the form

$$\mathbf{M}(\underline{\theta})\,\ddot{\underline{x}}\,(t,\underline{\theta}) + \mathbf{C}(\underline{\theta})\,\dot{\underline{x}}(t,\underline{\theta}) + \mathbf{K}(\underline{\theta})\,\underline{x}(t,\underline{\theta}) = \mathbf{R}^{*}(\underline{\theta})\,\underline{f}(t) \tag{1}$$

where $\underline{x}$ is the displacement response vector of dimension n, $\mathbf{M}(\underline{\theta})$, $\mathbf{C}(\underline{\theta})$ and $\mathbf{K}(\underline{\theta})$ are the mass, damping and stiffness matrices of dimension $n \times n$, $\underline{f}(t)$ is the excitation vector of dimension N_f, and $\mathbf{R}^{*}(\underline{\theta})$ is the $n \times N_f$-dimensional matrix that couples the excitation $\underline{f}(t)$ to the degrees of freedom of the structure. The

above mathematical model depends on a vector $\underline{\theta}$ comprised of all uncertain system parameters. In general, the matrices $\mathbf{M}(\underline{\theta})$, $\mathbf{K}(\underline{\theta})$, $\mathbf{C}(\underline{\theta})$ and $\mathbf{R}^*(\underline{\theta})$ are non-linear functions of the parameter set $\underline{\theta}$.

The formulation of the response of this system using modal analysis is next reviewed. In order to treat the general case of a non-classically damped structure, the equation of motion (1) is recast into the $2n$ state space form:

$$\dot{\underline{y}}\,(t,\underline{\theta}) + \mathbf{A}(\underline{\theta})\ \underline{y}(t,\underline{\theta}) = \mathbf{R}(\underline{\theta})\ \underline{f}(t) \tag{2}$$

where $\underline{y}^{\mathrm{T}} = [\underline{x}^{\mathrm{T}}, \dot{\underline{x}}^{\mathrm{T}}]$ is the $2n$-state vector, and

$$\mathbf{A}(\underline{\theta}) = \begin{bmatrix} \mathbf{0} & -\mathbf{I} \\ \mathbf{M}^{-1}(\underline{\theta})\mathbf{K}(\underline{\theta}) & \mathbf{M}^{-1}(\underline{\theta})\mathbf{C}(\underline{\theta}) \end{bmatrix}, \quad \mathbf{R}(\underline{\theta}) = \begin{Bmatrix} \underline{0} \\ \mathbf{M}^{-1}(\underline{\theta})\mathbf{R}^*(\underline{\theta}) \end{Bmatrix} \tag{3}$$

are the $2n \times 2n$ state matrix and the $2n \times N_f$ input matrix, respectively.

Denote by $\rho(t,\underline{\theta}) = \underline{\alpha}^{\mathrm{T}}(\underline{\theta})\ \underline{y}(t,\underline{\theta})$ a response quantity of interest, such as displacement, velocity, acceleration, or stress, given as a linear combination of the components of the response vector $\underline{y}(t,\underline{\theta})$. The dependence of $\underline{\alpha}(\underline{\theta})$ on $\underline{\theta}$ arises when, for example, the response quantity $\rho(t)$ is a stress or force component, and the modulus of elasticity involved in the computation of the stress or the force is assumed to be uncertain.

Let $\underline{\phi}^{(r)} \equiv \underline{\phi}^{(r)}(\underline{\theta})$ and $\underline{\psi}^{(r)} \equiv \underline{\psi}^{(r)}(\underline{\theta})$ be the rth eigenvectors of $\mathbf{A}(\underline{\theta})$ and $\mathbf{A}^{\mathrm{T}}(\underline{\theta})$, respectively, and $\lambda^{(r)} \equiv \lambda^{(r)}(\underline{\theta})$ be the rth eigenvalue of $\mathbf{A}(\underline{\theta})$. In structural dynamics applications, the matrix $\mathbf{A}$ in the system (2) is generally non-symmetric, and the associated eigenvalues and eigenvectors are all expected to occur in complex conjugate pairs. Physically, the complex eigenvalues are related to the frequencies and dampings of vibration of the system, and each complex conjugate pair corresponds to a mode of vibration of the structure. Let the modes of the system be arranged so that $\lambda^{(r+n)} = \overline{\lambda}^{(r)}$ is the complex conjugate of $\lambda^{(r)}$. Introducing the complex modal coordinate vector $\underline{\xi} = (\xi_1, \ldots, \xi_{2n})^{\mathrm{T}}$, the response $\rho(t)$ can be expressed as

$$\rho(t) = \sum_{r=1}^{2n} \xi_r(t) = 2\sum_{r=1}^{n} \mathrm{Re}\{\xi_r(t)\} \tag{4}$$

where the modal coordinate ξ_r satisfies the complex modal equation

$$\dot{\xi}_r + \lambda^{(r)}\xi_r = \underline{\beta}^{(r)\mathrm{T}}\ \underline{f}(t) \tag{5}$$

in which $\underline{\beta}^{(r)}$ is the participation factor vector given by $\underline{\beta}^{(r)\mathrm{T}} = 2\underline{\alpha}^{\mathrm{T}}\underline{\phi}_r\underline{\psi}_r^{\mathrm{T}}\mathbf{R}/(\underline{\psi}_r^{\mathrm{T}}\underline{\phi}_r)$. Each term in the second sum of (4) represents the contribution from the rth mode to the response quantity $\rho(t)$. Usually, in earthquake engineering applications, the number of modes N_m which contribute significantly to the response is small compared to the number of degrees of freedom, i.e. $N_m \ll n$. Also, only a very small number of degrees of freedom are excited externally, i.e. $N_f \ll n$.

Stochastic excitation

Consider the case of non-stationary zero-mean Gaussian stochastic processes used to model the uncertain temporal variability of the components of the excitation vector $\underline{f}(t)$. The response vector $\underline{y}(t,\underline{\theta})$ is also a zero-mean Gaussian process fully described by the covariance matrix $E[\underline{y}(t,\underline{\theta})\underline{y}^{\mathrm{T}}(t+\tau,\underline{\theta})]$ for any time lag τ,[19] where $E[\cdot]$ denotes mathematical expectation. A useful descriptor of the level of the response $\rho(t)$ is the variance $E[\rho^2(t)]$ which, by making use of equations (4) and (5), is given by

$$E[\rho^2(t,\underline{\theta})] = \sum_{r=1}^{2n}\sum_{s=1}^{2n} q_{rs}(t,\underline{\theta}) \tag{6}$$

where the quantities $q_{rs}(t,\underline{\theta}) = E[\xi_r(t,\underline{\theta})\xi_s(t,\underline{\theta})]$ are the modal cross-covariances at zero lag. Assuming that the excitation vector $\underline{f}(t)$ is a zero-mean Gaussian white noise process, it can be shown[19,20] that $q_{rs}(t,\underline{\theta})$

satisfies

$$\dot{q}_{rs}(t,\underline{\theta}) + \mu^{(rs)}(\underline{\theta})\, q_{rs}(t,\underline{\theta}) = \underline{\kappa}^{(rs)\mathrm{T}}(\underline{\theta})\, \underline{e}(t) \tag{7}$$

where

$$\mu^{(rs)}(\underline{\theta}) = \lambda^{(r)}(\underline{\theta}) + \lambda^{(s)}(\underline{\theta}) \tag{8}$$

and $\underline{\kappa}^{(rs)}(\underline{\theta})$ is a vector with the kth component

$$\kappa_k^{(rs)}(\underline{\theta}) = \beta_k^{(r)}(\underline{\theta})\, \beta_k^{(s)}(\underline{\theta}) \tag{9}$$

In deriving the form of the right-hand side of equation (7) it has been assumed, without loss of generality, that the force components $f_k(t)$, $k = 1,\ldots,N_f$ of the vector $\underline{f}(t)$ are independent, implying that $E[f_k(t)f_j(t)] = \delta_{kj}e_k(t)$, where $e_k(t)$ is the strength of the kth excitation $f_k(t)$, and δ_{kj} denotes the Kronecker delta. General stochastic excitation processes, modelled as the output of a linear filter with a Gaussian white noise input, can also be treated by analyzing an augmented linear system consisting of the original system and the input linear filter, and excited by a white noise process.

Equation (6) can be viewed as the modal expansion corresponding to the linear system of differential equations governing the evolution of the covariance response $E[\underline{y}(t,\underline{\theta})\underline{y}^{\mathrm{T}}(t,\underline{\theta})]$.[20] Equation (7) is the modal equation for the modal coordinate $q_{rs}(t,\underline{\theta})$. The analogy between the modal expansions (4) and (6) allows one to apply a methodology based on modal analysis to the stochastic excitation case for analyzing the system governing the evolution of the covariance response. Note that there are $(2n)^2$ modes involved in the modal expansion (6), instead of $(2n)$ modes involved in equation (4).

The simple algebraic expressions (8) and (9) can be used to obtain the eigenvalue $\mu^{(rs)}$ and modal participation factor $\underline{\kappa}^{(rs)}$ of the (rs) mode of equation (6) from the modal quantities corresponding to modes (r) and (s) of the original system (2). Because of the symmetry of q_{rs}, that is, $q_{rs} = q_{sr}$, only $n(2n+1)$ modal responses are distinct and, therefore, the number of modal components q_{rs} in equation (6) which need to be analyzed is significantly reduced. Recall that all eigenvalues $\lambda^{(r)}$, $r = 1,\ldots,2n$ appear in complex conjugate pairs, that is, $\lambda^{(r)} = \overline{\lambda}^{(r+n)}$, $r = 1,\ldots,n$. Then using equation (8), it is easy to verify that out of the $n(2n+1)$ quantities $\mu^{(rs)}$, n are real, given by $\lambda^{(r)} + \overline{\lambda}^{(r)}$, while the remaining $2n^2$ appear in complex conjugate pairs. Thus, similar to the case of equation (4), where only n out of $2n$ modes need to be analyzed, in the stochastic excitation case only n real and n^2 complex conjugate modes need to be analyzed. The interpretation of equations (6) and (7) as the modal superposition formula and modal equation, respectively, for the system of stochastic moments can be advantageously used to justify additional modal truncation techniques[20] which further reduce considerably the number of contributing modes in equation (6). Utilizing all the above observations, reduces significantly the computational effort involved in the analysis. This will be also seen later in a numerical example.

The zero-lag second-order statistics of a Gaussian stochastic process provide useful information about the level of system response and they can be used in analytical random vibration studies to compute mean rates of level crossing and exceedance probabilities which are useful in reliability studies.[19] More specifically, using Rice formula[21] and the fact that the response $\rho(t)$ is a zero-mean Gaussian process, the expected rate of up-crossing a given level b is given in terms of second-order statistics $\sigma_\rho^2 = E[\rho^2(t,\underline{\theta})]$, $\sigma_{\dot\rho}^2 = E[\dot\rho^2(t,\underline{\theta})]$ and $E[\rho(t,\underline{\theta})\dot\rho(t,\underline{\theta})]$ as

$$\nu^+(b,t,\underline{\theta}) = \left[\frac{\sigma_{\dot\rho}(1-s^2)^{1/2}}{2\pi\sigma_\rho}\right] \exp\left[\frac{-b^2}{2\sigma_\rho^2(1-s^2)}\right] + \left[\frac{sb\sigma_{\dot\rho}}{(2\pi)^{1/2}\sigma_\rho^2}\right] \exp\left[\frac{-b^2}{2\sigma_\rho^2}\right] \Phi\left[\frac{sb}{\sigma_\rho(1-s^2)^{1/2}}\right] \tag{10}$$

where $s = E[\rho(t,\underline{\theta})\dot\rho(t,\underline{\theta})]/(\sigma_\rho\sigma_{\dot\rho})$, and $\Phi(\cdot)$ is the Gaussian distribution function. The reliability function $L_g(b,t,\underline{\theta}) = \Pr[\max_{[0,t]} |\rho(t,\underline{\theta})| \leqslant b \mid \underline{\theta}]$, where $\Pr[A|B]$ denotes conditional probability of proposition A given proposition B, represents the probability that $\rho(t,\underline{\theta})$ has never reached the value b prior to time t for the given $\underline{\theta}$. For a high threshold level b, it can be assumed that the events of crossing such a level occur independently according to a Poisson process with mean rate $\nu^+(b,t,\underline{\theta})$, in which case the reliability function

can be approximated by[19]

$$L_g(b,t,\underline{\theta}) = L_g(b,0,\underline{\theta}) \exp\left[-2\int_0^t \nu^+(b,\tau,\underline{\theta})\,d\tau\right] \tag{11}$$

For zero initial conditions with probability one, $L_g(b,0,\underline{\theta}) = 1$ for any $b > 0$ and any $\underline{\theta}$. The subscript g in L_g is used to indicate that the stochastic response $\rho(t,\underline{\theta})$ is Gaussian.

ANALYSIS OF STRUCTURES WITH UNCERTAIN PROPERTIES

Modelling of system uncertainties

Consider the vector $\underline{\theta}$ comprised of all the uncertain system parameters. The uncertainties are quantified by modelling these parameters by random variables with specified probability distribution representing the relative plausibilities of the possible values of these parameters. In past studies, the uncertainty in the mass, damping and stiffness matrices of a structure has been parameterized in the linear form[6,7,15,16,18,22]

$$\mathbf{D}(\underline{\theta}) = \mathbf{D}_0 + \sum_{i=1}^{N_\theta} \mathbf{D}_i \theta_i \tag{12}$$

where the matrix $\mathbf{D}(\underline{\theta})$ represents any of $\mathbf{K}(\underline{\theta})$, $\mathbf{C}(\underline{\theta})$ or $\mathbf{M}(\underline{\theta})$. The parameters θ_i are chosen to be dimensionless. The selection of matrices $\mathbf{D}_0$ and $\mathbf{D}_i$ is specified uniquely if certain assumptions are made regarding the statistical properties of the θ_is. If it is assumed that $E[\theta_i] = 0$, $i = 1,\ldots,N_\theta$, then $\mathbf{D}_0$ represents the expected value of $\mathbf{D}(\underline{\theta})$. Also, by assuming $E[\theta_i^2] = 1$, the scaling of the contribution $\mathbf{D}_i$, due to the parameters θ_i, to $\mathbf{D}(\underline{\theta})$ is uniquely specified.

Equation (12) also arises in stochastic finite element formulations which employ stochastic fields to model the spatial variation of various uncertain structural properties, such as modulus of elasticity, soil stiffness, mass or density. Applying existing random field discretization techniques,[7,8,18,22] the uncertainty associated with continuous properties can be described by a set of correlated discrete random variables. Finally, making use of suitable spectral decomposition methods, a set $\underline{\theta}$ of uncorrelated random variables can be derived to describe the system uncertainties.[9,18,23]

It should be noted that often a linear parameterization of the structural properties $\mathbf{M}(\underline{\theta})$, $\mathbf{C}(\underline{\theta})$ and $\mathbf{K}(\underline{\theta})$ as a function of uncertain parameters $\underline{\theta}$ is not representative of situations encountered in practical applications. For example, uncertainties concerning the damping of a system are usually specified as uncertainty in the modal damping coefficients. Treating the damping coefficients as uncertain and including them in the vector $\underline{\theta}$, results in the damping matrix $\mathbf{C}(\underline{\theta})$ being a nonlinear function of $\underline{\theta}$. Unless Rayleigh damping $\mathbf{C} = \alpha\mathbf{M} + \beta\mathbf{K}$ with deterministic coefficients α and β is assumed,[24] the damping matrix of the system does not in general admit the simple representation (12). Using, for example, Rayleigh damping with uncertain coefficients α and β, included in the parameter vector $\underline{\theta}$, results in a nonlinear dependence of the damping matrix on the system uncertain variables $\underline{\theta}$, instead of the linear one given by equation (12). The method that is proposed in this paper is capable of treating non-linear matrix functions $\mathbf{M}(\underline{\theta})$, $\mathbf{C}(\underline{\theta})$ and $\mathbf{K}(\underline{\theta})$.

Quantification of response of uncertain structures

The response $\rho(t,\underline{\theta})$ of uncertain structures subjected to deterministic excitations are quantified herein by statistical descriptors such as mean response $m_\rho(t) = E_{\underline{\theta}}[\rho(t,\underline{\theta})]$ describing the expected level of the response, and standard deviation of response $\sigma_\rho(t) = (E_{\underline{\theta}}[\rho^2(t,\underline{\theta})] - m_\rho^2(t))^{1/2}$ describing the deviation of the response about the mean level. To compute these response descriptors from the probability distribution assigned to the uncertain parameters $\underline{\theta}$, one needs to know the response variability due to the variability of the parameters $\underline{\theta}$. The response variability could also be used to compute other response descriptors such as exceedance probabilities useful in reliability analysis.

For uncertain structures subjected to stochastic excitation, the same statistical descriptors, such as mean and variance of the response can be used. However, the expectations needed in the computation of the mean

and variance have to be carried out over both the loading and system uncertainties. It is usually convenient to first compute the mean $m_\rho(t,\underline{\theta}) = E[\rho(t,\underline{\theta})]$ and variance $\sigma_\rho^2(t,\underline{\theta}) = E[\rho^2(t,\underline{\theta})] - m_\rho^2(t,\underline{\theta})$ of the response conditional on the value of $\underline{\theta}$. The total probability law can then be used to compute the unconditional mean $m_\rho(t)$ and variance $\sigma_\rho^2(t)$ of the response as follows:

$$m_\rho(t) = E_{\underline{\theta}}[m_\rho(t,\underline{\theta})] \tag{13}$$

$$\sigma_\rho^2(t) = E_{\underline{\theta}}[\sigma_\rho^2(t,\underline{\theta})] + \mathrm{Var}_{\underline{\theta}}[m_\rho(t,\underline{\theta})] \tag{14}$$

For zero-mean Gaussian excitations assumed in the present study, the mean response is zero, that is, $m_\rho(t) = 0$ since $m_\rho(t,\underline{\theta}) = 0$ for every $\underline{\theta}$. Also, in this case, the variance is given by

$$\sigma_\rho^2(t) = E_{\underline{\theta}}[\sigma_\rho^2(t,\underline{\theta})] = \int_{\underline{\theta}} \sigma_\rho^2(t,\underline{\theta})\, p(\underline{\theta})\, \mathrm{d}\underline{\theta} \tag{15}$$

An important difference between a deterministic and an uncertain system subjected to the same Gaussian stochastic excitation is that the response of the uncertain system is not a Gaussian process. Some measures of non-Gaussian processes, important in reliability studies,[25] are the skewness and the coefficient of excess (COE). These measures are both zero for a Gaussian process. For the uncertain linear system driven by zero-mean Gaussian loads, the skewness is zero. However, the COE defined in terms of the second and fourth moments as

$$\mathrm{COE}(t) = \frac{E[\rho^4(t)]}{E^2[\rho^2(t)]} - 3 = \frac{E_{\underline{\theta}}[E[\rho^4(t,\underline{\theta})]]}{E_{\underline{\theta}}^2[\sigma_\rho^2(t,\underline{\theta})]} - 3 \tag{16}$$

is non-zero. Specifically, since the response process conditional on $\underline{\theta}$ is Gaussian, it follows that $E[\rho^4(t,\underline{\theta})] = 3\sigma_\rho^4(t,\underline{\theta})$, which yields

$$\mathrm{COE}(t) = 3\frac{\mathrm{Var}_{\underline{\theta}}[\sigma_\rho^2(t,\underline{\theta})]}{E_{\underline{\theta}}^2[\sigma_\rho^2(t,\underline{\theta})]} = 3\left(\mathrm{Cov}_{\underline{\theta}}[\sigma_\rho^2(t,\underline{\theta})]\right)^2 \tag{17}$$

where $\mathrm{Cov}_{\underline{\theta}}$ denotes coefficient of variation with respect to $\underline{\theta}$. The fact that COE is always positive means that more probability mass is in the tails of the distribution. The implication of this in reliability is that the probability of failure, measured in terms of the probability of exceeding a given level, increases for the uncertain system. Thus, neglecting the non-Gaussian effects of the response may lead to unconservative reliability estimates.

The random vibration results for computing the probabilities of level exceedance can be extended to the case of uncertain structures. The reliability function $L(b,t) = \Pr[\max_{[0,t]} |\rho(t,\underline{\theta})| \leqslant b]$, which is defined as the probability that $\rho(t,\underline{\theta})$ has never reached the value b prior to time t for all possible values of $\underline{\theta}$, can be shown to be

$$L(b,t) = E_{\underline{\theta}}[L_{\mathrm{g}}(b,t,\underline{\theta})] = \int_{\underline{\theta}} L_{\mathrm{g}}(b,t,\underline{\theta})\, p(\underline{\theta})\, \mathrm{d}\underline{\theta} \tag{18}$$

where $L_{\mathrm{g}}(b,t,\underline{\theta})$ is the conditional reliability function (11).

It is worth noting that the computation of $\sigma_\rho^2(t)$ and $\mathrm{COE}(t)$ involves the computation of the mean and variance over $\underline{\theta}$ of the conditional quantity $\sigma_\rho^2(t,\underline{\theta})$. Moreover, $L(b,t)$ is the mean over $\underline{\theta}$ of the conditional quantity $L_{\mathrm{g}}(b,t,\underline{\theta})$. Thus, an important first step in computing the response descriptors is to model the variability with respect to $\underline{\theta}$ of the conditional second-order statistics of the response $\rho(t,\underline{\theta})$. Once this variability is known, integration over the domain of $\underline{\theta}$ can be carried out numerically or using simulations in order to compute unconditional response descriptors.

Numerical integration or simulations methods are powerful in estimating the aforementioned response descriptors in both the deterministic and stochastic excitation cases. However, if applied directly using the mathematical model, they require a large number of repeated structural analyses to be performed for different

values of $\underline{\theta}$. Next, an accurate and computationally efficient method of analysis is proposed for approximating the variability of the response by simplified expressions. These approximate expressions for the variability can then be successfully used to derive analytical expressions for computing $m_p(t)$, $\sigma_p^2(t)$ and COE(t), allowing for a very efficient computation of these response descriptors.

MODAL VARIABILITY

Variability in eigenvalues and participation factors

The modelling of the variability of the eigenvalue $\lambda^{(r)}$ and the participation factor vector $\underline{\beta}^{(r)}$, appearing in modal equation (5), in terms of the variability of the uncertain parameters is first addressed. Such modelling will be utilized in the subsequent section to derive analytical expressions for the response variability. The variability of the modal quantities $\lambda^{(r)} \equiv \lambda^{(r)}(\underline{\theta})$ and $\underline{\beta}^{(r)} \equiv \underline{\beta}^{(r)}(\underline{\theta})$, $r = 1,\ldots,N_m$ is implicitly defined by solving the eigenvalue problem for $\mathbf{A}(\underline{\theta})$. However, to enable an analytical formulation of the modal response variability, the multivariable quadratic polynomial expansions

$$\lambda^{(r)}(\underline{\theta}) = \lambda_0^{(r)} + \sum_{i=1}^{N_\theta} \lambda_i^{(r)}\theta_i + \sum_{i=1}^{N_\theta}\sum_{j=i}^{N_\theta} \lambda_{ij}^{(r)}\theta_i\theta_j + \varepsilon_\lambda^{(r)}(\underline{\theta}), \tag{19}$$

and

$$\underline{\beta}^{(r)}(\underline{\theta}) = \underline{\beta}_o^{(r)} + \sum_{i=1}^{N_\theta} \underline{\beta}_i^{(r)}\theta_i + \sum_{i=1}^{N_\theta}\sum_{j=i}^{N_\theta} \underline{\beta}_{ij}^{(r)}\theta_i\theta_j + \underline{\varepsilon}_\beta^{(r)}(\underline{\theta}), \quad k = 1,\ldots,N_f \tag{20}$$

$r = 1,2,\ldots,N_m$ are utilized to explicitly express the modal quantities $\lambda^{(r)}(\underline{\theta})$ and $\underline{\beta}^{(r)}(\underline{\theta})$ as a function of the uncertain variables $\theta_1,\theta_2,\ldots,\theta_{N_\theta}$ over the domain of interest, that is, over the range of $\underline{\theta}$ values where the probability density function $p(\underline{\theta})$ is significant. The error terms $\varepsilon_\lambda^{(r)}(\underline{\theta})$ and $\underline{\varepsilon}_\beta^{(r)}(\underline{\theta})$ model the difference between the values predicted by the previous multivariable expansions and the exact values obtained by solving the eigenvalue problem. The $N_c = [N_\theta(N_\theta+1)]/2+N_\theta+1$ unknown coefficients of each expansion, arranged in vectors $\underline{\hat{\lambda}}^{(r)\mathrm{T}} = [\lambda_0^{(r)}, \lambda_1^{(r)},\ldots,\lambda_{N_\theta}^{(r)},\lambda_{11}^{(r)},\lambda_{12}^{(r)},\ldots,\lambda_{N_\theta N_\theta}^{(r)}]$ and $\underline{\hat{\beta}}^{(r)\mathrm{T}} = [\underline{\beta}_0^{(r)\mathrm{T}}, \underline{\beta}_1^{(r)\mathrm{T}},\ldots,\underline{\beta}_{N_\theta}^{(r)\mathrm{T}}, \underline{\beta}_{11}^{(r)\mathrm{T}}, \underline{\beta}_{12}^{(r)\mathrm{T}},\ldots,\underline{\beta}_{N_\theta N_\theta}^{(r)\mathrm{T}}]$, are determined by minimizing the errors in a least-squares sense, that is, by minimizing $\sum_k[\varepsilon_\lambda^{(r)}(\underline{\theta}_k)]^2$ and $\sum_k[\varepsilon_{\beta,l}^{(r)}(\underline{\theta}_k)]^2$, where $\underline{\theta}_k$ are appropriately selected interpolation points in the N_θ-dimensional space of the parameters $\underline{\theta}$. The accuracy of the fit will depend on how closely the exact surface that is being fitted resembles a quadratic surface, as well as on the number of points used in the fit. In the present analysis, the following N_c interpolation points are chosen: $\underline{\theta} = \underline{0}$, $\underline{\theta} = \pm\underline{\alpha}_i$, $i = 1,\ldots,N_\theta$, and $\underline{\theta} = \underline{\alpha}_i + \underline{\alpha}_j$, $i = 1,\ldots,N_\theta$, $j = 1,\ldots,N_\theta$, where $\underline{\alpha}_i = \varepsilon_i(\delta_{1i},\delta_{2i},\ldots,\delta_{N_\theta i})^{\mathrm{T}}$, ε_i defines the distance of the interpolation points from the origin, and δ_{ij} is the Kronecker delta. The value of ε_i depends on the distribution assumed for θ_i.[5]

The estimation of the coefficients of the quadratic expansions (19) and (20) requires N_c repeated eigenvalue analyses of deterministic systems corresponding to different $\underline{\theta}_k$ values. For large systems, this requires considerable computational effort. Substantial savings are achieved by using techniques that efficiently compute the eigensolutions for the first few significant structural modes, avoiding repeated analyses corresponding to a large number of different system configurations. Modal condensation-type approaches[26] or subspace iteration methods allow for directly solving for the eigenproperties of a modified structure ($\underline{\theta} \neq \underline{0}$) using the eigenproperties of the base structure ($\underline{\theta} = \underline{0}$), hence avoiding a complete reanalysis.

The quadratic expansions (19) and (20) may in some cases be replaced by cubic expansions. The latter were found to provide a better fit for certain type of problems. The estimation of the coefficients of the cubic expansion is done following a procedure similar to that described for the quadratic one.

In the stochastic excitation case, the polynomial expansions for $\mu^{(rs)}(\underline{\theta}) = \lambda^{(r)}(\underline{\theta}) + \lambda^{(s)}(\underline{\theta})$ and $\kappa_k^{(rs)}(\underline{\theta}) = \beta_k^{(r)}(\underline{\theta})\beta_k^{(s)}(\underline{\theta})$ can be based on the expansions derived for $\lambda^{(r)}(\underline{\theta})$ and $\beta_k^{(r)}(\underline{\theta})$. The order of the expansion for the eigenvalue $\mu^{(rs)}(\underline{\theta})$ is exactly the same as the order used for the corresponding $\lambda^{(r)}(\underline{\theta})$ and $\lambda^{(s)}(\underline{\theta})$.

It is also obvious that the coefficients $\mu_o^{(rs)}(\underline{\theta})$, $\mu_i^{(rs)}(\underline{\theta})$ and $\mu_{ij}^{(rs)}(\underline{\theta})$ corresponding to a quadratic polynomial fit for $\mu^{(rs)}(\underline{\theta})$ similar to (19), are simply given by $\mu_o^{(rs)}(\underline{\theta}) = \lambda_o^{(r)}(\underline{\theta}) + \lambda_o^{(s)}(\underline{\theta})$, $\mu_i^{(rs)}(\underline{\theta}) = \lambda_i^{(r)}(\underline{\theta}) + \lambda_i^{(s)}(\underline{\theta})$ and $\mu_{ij}^{(rs)}(\underline{\theta}) = \lambda_{ij}^{(r)}(\underline{\theta}) + \lambda_{ij}^{(s)}(\underline{\theta})$. Note that these values minimize $\sum_k [\varepsilon_\mu^{(rs)}(\underline{\theta}_k)]^2$, where $\varepsilon_\mu^{(rs)}(\underline{\theta}_k)$ represents the error of the quadratic expansion for $\mu^{(rs)}(\underline{\theta})$ at the interpolation points.

The order of the polynomial approximation for the factors $\kappa_k^{(rs)}(\underline{\theta})$ is the product of the orders of the expansions used for $\beta_k^{(r)}(\underline{\theta})$ and $\beta_k^{(s)}(\underline{\theta})$. If the quadratic expansion (20) is used, then the expansion for $\kappa_k^{(rs)}(\underline{\theta})$ is of fourth order, and its coefficients can be easily obtained from the coefficients of the quadratic expansions (20). However, these coefficients are not optimal in the least-squares sense. A fourth-order polynomial expansion for $\kappa_k^{(rs)}(\underline{\theta})$ based on least-squares fit would be more accurate. Moreover, numerical results have shown that even a cubic or quadratic expansion for $\kappa_k^{(rs)}(\underline{\theta})$ can, in some cases, result in reasonable accuracy provided that these expansions are obtained by the least-squares fit method. In order to estimate the coefficients of the cubic or fourth-order polynomial expansions for $\kappa_k^{(rs)}(\underline{\theta})$ based on a least-squares method, more eigenvalue problems have to be solved at additional points $\underline{\theta}_k$, appropriately selected in the N_θ-dimensional space of the parameters $\underline{\theta}$. This can be done efficiently by utilizing the relatively accurate quadratic polynomial approximation for $\lambda^{(r)}$ to provide estimates of the eigenvalues for any additional point $\underline{\theta}_k$, thus avoiding the solution of the computationally expensive eigenvalue problems. Then, using these estimates for $\lambda^{(r)}(\underline{\theta}_k)$, the eigenvectors $\underline{\phi}(\underline{\theta}_k)$ and $\underline{\psi}(\underline{\theta}_k)$ involved in computing $\underline{\beta}^{(r)}(\underline{\theta}_k)$ and then $\underline{\kappa}^{(rs)}(\underline{\theta}_k)$ can be obtained by solving the linear systems $\mathbf{A}(\underline{\theta}_k)\underline{\phi}(\underline{\theta}_k) = \lambda^{(r)}(\underline{\theta}_k)\underline{\phi}(\underline{\theta}_k)$ and $\mathbf{A}^{\mathrm{T}}(\underline{\theta}_k)\underline{\psi}(\underline{\theta}_k) = \lambda^{(r)}(\underline{\theta}_k)\underline{\psi}(\underline{\theta}_k)$.

Perturbation techniques could also be employed to explicitly describe the variability of the modal properties for small variations of the system parameters.[27] However, methods based on least-squares fit estimates of the quadratic polynomial approximations (19) and (20) were found to provide more accurate estimates of the variability of each eigenproperty over the range of variation of the system parameters corresponding to significant probability.[5]

Modal response variability

The formulation for the variability of the modal response $\xi_r(t, \underline{\theta})$ is mainly presented for the case of deterministic excitation. Because of the similarity of the modal equations (5) and (7), the formulation will also be applicable to the stochastic excitation case for computing the variability of the modal responses $q_{rs}(t, \underline{\theta})$. Highly nonlinear dependencies of the modal response on the system variables observed for dynamical systems[6,15,16] cannot be adequately modelled using perturbation or second-moment analysis,[10,14,23,28] even if the uncertainties in the system parameters are small. Alternative methods, based on expansions of the modal response into an orthogonal set of functions involving the system's uncertain variables,[1,6,9,16,18] are best suited for providing accurate estimates of the response variability for dynamic systems.

Specifically, let $P_{l_p}(\theta_p)$, $l_p = 0, 1, 2, \ldots, \infty$ denote a set of orthogonal polynomials of one variable θ_p satisfying the orthogonality relationship

$$\int_{\theta_p} w(\theta_p) P_{l_p}(\theta_p) P_{l'_p}(\theta_p)\, \mathrm{d}\theta_p = a_{l_p} \delta_{l_p l'_p}; \quad p = 1, 2, \ldots, N_\theta, \tag{21}$$

where l_p is the degree of $P_{l_p}(\theta_p)$, $w(\theta_p)$ is a weighting function, and a_{l_p} is a constant dependent upon the given set of orthogonal polynomials. Considering the orthogonal basis formed by the set of multivariable polynomials $\prod_{p=1}^{N_\theta} P_{l_p}(\theta_p)$, where l_i is non-negative integer and $i = 1, \ldots, N_\theta$, the complex-valued modal response $\xi^{(r)}(t, \underline{\theta})$ defined by (5) can be approximated by the truncated series expansion

$$\{\xi^{(r)}(t, \underline{\theta})\} \simeq \sum_{0 \leqslant L \leqslant N_e^{(r)}} \chi^{(r)}_{l_1 l_2 \ldots l_{N_\theta}}(t) \prod_{p=1}^{N_\theta} P_{l_p}(\theta_p); \quad r = 1, 2, \ldots, N_m, \tag{22}$$

where $L = \sum_{p=1}^{N_\theta} l_p$, $N_e^{(r)}$ is the order of approximation for mode r in the space of the variables $\underline{\theta}$, and $[\chi^{(r)}_{l_1 l_2 \ldots l_{N_\theta}}(t)]_{0 \leqslant L \leqslant N_e^{(r)}}$ are unknown complex-valued coefficients which are the elements of the N_θ-dimensional

tensor $\chi^{(r)}_{l_1 l_2 \ldots l_{N_\theta}}$ satisfying $0 \leqslant L \leqslant N_e^{(r)}$. The response variability of the rth mode is fully characterized by the coefficients $[\chi^{(r)}_{l_1 \ldots l_{N_\theta}}(t)]_{0 \leqslant L \leqslant N_e^{(r)}}$ of the orthogonal polynomial expansion (22). The number of these unknown coefficients is

$$N_d^{(r)} = \frac{(N_e^{(r)} + N_\theta)!}{N_e^{(r)}!\ N_\theta!} \tag{23}$$

The weighted residual method can be used to derive a set of equations for computing the unknown coefficients in the expansion (22). The following general recursive relationships[29] are useful in the development of such equations:

$$(\theta_p)^k P_{l_p}(\theta_p) = \sum_{i=l_p-k}^{l_p+k} C_k(i, l_p) P_i(\theta_p), \quad k = 1, 2, \ldots, z \tag{24}$$

where the value of $C_k(i, l_p)$ depends on the chosen orthogonal set of polynomials $P_{l_p}(\theta_p)$. Substituting the orthogonal expansion (22) and the expansions (19) and (20) into the modal equation (5), multiplying the resulting equation by $\prod_{p=1}^{N_\theta} w(\theta_p) P_{l'_p}(\theta_p)$, integrating over the θ_p's, and finally making use of the recursive formulas (24) and the orthogonality condition (21), one obtains the following system of first-order ordinary differential equations for the unknown coefficients $\chi^{(r)}_{l_1 \ldots l_{N_\theta}}(t)$ corresponding to mode r:

$$\begin{aligned}
&\dot{\chi}^{(r)}_{l_1 \cdots l_{N_\theta}} + \lambda_0^{(r)}\, \chi^{(r)}_{l_1 \cdots l_{N_\theta}} + \sum_{i=1}^{N_\theta} \lambda_i^{(r)} \sum_{\ell_i = l_i - 1}^{l_i + 1} d_i(\underline{l})\, \chi^{(r)}_{l_1 \cdots \ell_i \cdots l_{N_\theta}} \\
&\quad + 2 \sum_{i=1}^{N_\theta} \sum_{j=1}^{N_\theta} \lambda_{ij}^{(r)} \sum_{\ell_i = l_i - 1}^{l_i + 1} \sum_{\ell_j = l_j - 1}^{l_j + 1} d_{ij}(\underline{l})\, \chi^{(r)}_{l_1 \cdots \ell_i \cdots \ell_j \cdots l_{N_\theta}} + \sum_{i=1}^{N_\theta} \lambda_{ii}^{(r)} \sum_{\ell_i = l_i - 2}^{l_i + 2} d_{ii}(\underline{l}) \chi^{(r)}_{l_1 \cdots \ell_i \cdots l_{N_\theta}} \\
&\quad = \hat{h} \left[h_0(\underline{l}) \underline{\beta}_0^{(r)\mathrm{T}} + \sum_{i=1}^{N_\theta} h_i(\underline{l}) \underline{\beta}_i^{(r)\mathrm{T}} + 2 \sum_{i=1}^{N_\theta} \sum_{j=1}^{N_\theta} h_{ij}(\underline{l}) \underline{\beta}_{ij}^{(r)\mathrm{T}} + \sum_{i=1}^{N_\theta} h_{ii}(\underline{l}) \underline{\beta}_{ii}^{(r)\mathrm{T}} \right] \underline{f}(t), \\
&\quad l_m = 0, 1, \ldots, \quad m = 1, 2, \ldots, N_\theta
\end{aligned} \tag{25}$$

where the expressions for the coefficients $d_i(\underline{l})$, $d_{ij}(\underline{l})$, $h_0(\underline{l})$, $h_i(\underline{l})$, $h_{ij}(\underline{l})$ and $\hat{h}$ are given in the Appendix. These coefficients depend only on the polynomial basis used in the expansion (22) and are independent of the modal properties. Rearranging the elements $[\chi^{(r)}_{l_1 \ldots l_{N_\theta}}]_{0 \leqslant L \leqslant N_e^{(r)}}$ into an $N_d^{(r)}$-dimensional vector $\underline{z}^{(r)}$, the evolution of $\underline{z}^{(r)}$ is governed by the set of equations

$$\dot{\underline{z}}^{(r)}(t) + \boldsymbol{\Gamma}(\hat{\underline{\lambda}}^{(r)})\, \underline{z}^{(r)}(t) = \boldsymbol{\Upsilon}(\hat{\underline{\beta}}^{(r)}) \underline{f}(t) \tag{26}$$

in which the $N_d^{(r)} \times N_d^{(r)}$ matrix $\boldsymbol{\Gamma}(\hat{\underline{\lambda}}^{(r)})$ and the $N_d^{(r)} \times N_f$ matrix $\boldsymbol{\Upsilon}(\hat{\underline{\beta}}^{(r)})$ are functions of the complex-valued coefficients $\hat{\underline{\lambda}}^{(r)}$ and $\hat{\underline{\beta}}^{(r)}$ involved in the quadratic expansions (19) and (20). The form of $\boldsymbol{\Gamma}(\hat{\underline{\lambda}}^{(r)})$ and $\boldsymbol{\Upsilon}(\hat{\underline{\beta}}^{(r)})$ can be easily obtained by comparing equations (25) and (26). Both $\boldsymbol{\Gamma}(\hat{\underline{\lambda}}^{(r)})$ and $\boldsymbol{\Upsilon}(\hat{\underline{\beta}}^{(r)})$ satisfy the linearity properties $\boldsymbol{\Gamma}(\underline{\lambda}_1 + \mathrm{i}\underline{\lambda}_2) = \boldsymbol{\Gamma}(\underline{\lambda}_1) + \mathrm{i}\boldsymbol{\Gamma}(\underline{\lambda}_2)$ and $\boldsymbol{\Upsilon}(\underline{\beta}_1 + \mathrm{i}\underline{\beta}_2) = \boldsymbol{\Upsilon}(\underline{\beta}_1) + \mathrm{i}\boldsymbol{\Upsilon}(\underline{\beta}_2)$, respectively, which are useful in writing the complex-valued system (26) of $N_d^{(r)}$ equations as a real-valued system of $2N_d^{(r)}$ first-order differential equations for the real and imaginary parts of the coefficients $[\chi^{(r)}_{l_1 l_2 \ldots l_{N_\theta}}]_{0 \leqslant L \leqslant N_e^{(r)}}$. Available numerical algorithms are used to solve the resulting system of $2N_d^{(r)}$ real differential equations.

It should be noted that the dimension $N_d^{(r)}$ of the matrix $\boldsymbol{\Gamma}(\hat{\underline{\lambda}}^{(r)})$ increases fast with the number of uncertain variables involved in the problem, which may pose computer storage problems. However, the matrix $\boldsymbol{\Gamma}(\hat{\underline{\lambda}}^{(r)})$

is sparse which reduces both the computational effort and computer storage requirements. In particular, the lower the order of the polynomial expansion for $\underline{\lambda}^{(r)}$, the more sparse the matrix $\mathbf{\Gamma}(\widehat{\underline{\lambda}}^{(r)})$ is. Also, the fact that some of the coefficients $C_k(i,j)$, $i = j-k,\ldots,j+k$ have zero values[29] reduces further the number of non-zero elements in $\mathbf{\Gamma}(\widehat{\underline{\lambda}}^{(r)})$. The order of the polynomial expansion for $\underline{\beta}^{(r)}$ has no effect on the sparseness of the matrix $\mathbf{\Gamma}(\widehat{\underline{\lambda}}^{(r)})$ and can be shown to have negligible effect on the total computational requirements.

Assuming that the system matrices $\mathbf{M}(\underline{\theta})$, $\mathbf{C}(\underline{\theta})$ and $\mathbf{K}(\underline{\theta})$ admit the linear representation (12), Jensen and Iwan[1,6,18] have applied the weighted residual method to obtain the variability of the system responses directly from the system (2) of $2n$ equations. However, this results in a system of $2nN_d$ coupled differential equations for the unknown coefficients of the orthogonal expansion, where, in order to achieve the same accuracy as with the proposed method, N_d must be equal to $\max\{N_d^{(1)},\ldots,N_d^{(N_m)}\}$. An advantage of the proposed method is that each mode can be treated separately, resulting to at most $2N_d^{(r)}$ coupled differential equations for mode r. Moreover, it was found that lower modes corresponding to low eigenvalues can yield accurate results using lower-order truncation $N_e^{(r)}$ in the expansion (22). For systems with many degrees of freedom (large n), the proposed approach based on modal expansion is computationally more efficient than other OEM methods, especially when only a few modes contribute significantly to the response, that is, when $N_m \ll n$. For the cases where many of the system modes need to be included in the analysis, the proposed method will require large computational effort in evaluating the eigenvalues and eigenvectors for all modes. However, other OEM approaches[1,6,18] based on expansions of the $2n$-dimensional state vector $\underline{y}$ directly, instead of the scalar modal responses ξ_r, are likely to suffer from excessive storage requirements when n is large.

EXPRESSIONS FOR RESPONSE VARIABILITY AND STATISTICS

Substituting equation (22) for the modal response variability into the mode superposition formula (4) and interchanging the order of summation, the response $\rho(t,\underline{\theta})$ in terms of $\underline{\theta}$ is given by

$$\rho(t,\underline{\theta}) = \sum_{0 \leqslant L \leqslant N_e} \chi_{l_1 \ldots l_{N_\theta}}(t) \prod_{p=1}^{N_\theta} P_{l_p}(\theta_p), \tag{27}$$

where $N_e = \max\{N_e^{(1)},\ldots,N_e^{(N_m)}\}$,

$$\chi_{l_1 \ldots l_{N_\theta}}(t) = \sum_{r=1}^{N_m} \chi^{(r)}_{l_1 \ldots l_{N_\theta}}(t) \tag{28}$$

and $[\chi^{(r)}_{l_1 \ldots l_{N_\theta}}]_{N_e^{(r)} < L \leqslant N_e} = 0$. The coefficients $\chi_{l_1 \ldots l_{N_\theta}}(t)$, which fully characterize the response variability, are obtained as a linear superposition of the coefficients $\chi^{(r)}_{l_1 \ldots l_{N_\theta}}(t)$ for each mode. Equation (27) extends the results derived by Iwan and Jensen[1,6] to the case where the system is described by its modes.

The expansion (27) is well suited for deriving simple analytical expressions for the statistical moments of the response by choosing the polynomials $P_{l_p}(\theta_p)$ so that the orthogonality property (21) is taken with respect to the mean operation, that is, the weighting function $w(\theta_p)$ in (21) equals the probability density function $p(\theta_p)$ of the random variable θ_p. Two examples of polynomial bases $P_{l_p}(\theta_p)$ that satisfy the above condition are the Hermite and Legendre polynomials for normally and uniformly distributed random variable θ_p, respectively. The values of a_{l_p} involved in the orthogonality condition (21) are $a_{l_p} = 2^{l_p}(l_p)!/\sqrt{2}$ and $a_{l_p} = 2/(2l_p+1)$ for the Hermite and Legendre polynomials, respectively.[29] Under such conditions, it can be shown that the mean $m_\rho(t)$ and variance $\sigma_\rho^2(t)$ of the response take the simple forms

$$m_\rho(t) = E_{\underline{\theta}}[\rho(t,\underline{\theta})] = (a_0)^{N_\theta} \chi_{0 \ldots 0}(t) \tag{29}$$

$$\sigma_\rho^2(t) = E_{\underline{\theta}}[\rho^2(t,\underline{\theta})] - m_\rho^2(t) = \sum_{1 \leqslant L \leqslant N_p} [\chi_{l_1 \ldots l_{N_\theta}}(t)]^2 \prod_{p=1}^{N_\theta} a_{l_p} \tag{30}$$

It is worth noting that the mean response depends only on the zeroth coefficient $\chi_{0\ldots0}(t)$ of the orthogonal expansion. In several cases, accurate values of the zeroth coefficient can be obtained by solving the system (26) with order of truncation as low as $N_e^{(r)} = 1$.

For other common types of probability distributions, such as parabolic, triangular or gamma distributions, for which a set of polynomials satisfying the orthogonality condition (21) with $w(\theta_p) = p(\theta_p)$ is not available, or for cases where the original parameters are not independent, appropriate nonlinear probability transformations can be used to transform the system of random variables into a new set of statistically independent, standard normal or uniform random variables.[30] In this case, non-linear relationships between the system mass, stiffness and damping matrices and the new random variables are obtained as opposed to the linear relationship (12) used by other methods. In contrast to other OEM methods,[1,6,9,18] the present methodology, which was developed for general non-linear functions $\mathbf{M}(\underline{\theta})$, $\mathbf{C}(\underline{\theta})$ and $\mathbf{K}(\underline{\theta})$, can efficiently treat any probability distribution assumed for the uncertain system variables.

It should be mentioned that once the coefficients $[\chi^{(r)}_{l_1\ldots l_{N_\theta}}]_{0\leqslant L\leqslant N_e^{(r)}}$ have been obtained, equation (27) can be used in simulation studies, for example in calculating reliability, to provide estimates of the response $\rho(t,\underline{\theta})$ for a large number of points $\underline{\theta}$ by avoiding repeated and often expensive analyses of structural response for each $\underline{\theta}$.

Finally, for the stochastic excitation case, the variability of the conditional mean-square response $\sigma_\rho^2(t,\underline{\theta}) = E[\rho^2(t,\underline{\theta})]$ in terms of $\underline{\theta}$ is given by equation (27) with $\chi_{l_1\ldots l_{N_\theta}}(t)$ replaced by

$$\chi_{l_1\ldots l_{N_\theta}}(t) = \sum_{r=1}^{N_m}\sum_{s=1}^{N_m} \chi^{(rs)}_{l_1\ldots l_{N_\theta}}(t) \tag{31}$$

where, in this case, the coefficients $\chi^{(rs)}_{l_1\ldots l_{N_\theta}}(t)$ satisfy equation (26) with $\widehat{\underline{\lambda}}^{(r)}$ replaced by $\widehat{\underline{\mu}}^{(rs)}$ and $\widehat{\underline{\beta}}_k^{(r)}$ replaced by $\widehat{\underline{\kappa}}_k^{(rs)}$. The unconditional mean-square response and its variance, also needed in equation (17) to calculate the COE(t), are respectively given by

$$\sigma_\rho^2(t) = E_{\underline{\theta}}[\sigma_\rho^2(t,\underline{\theta})] = (a_0)^{N_\theta}\chi_{0\ldots0}(t) \tag{32}$$

$$\mathrm{Var}_{\underline{\theta}}[\sigma_\rho^2(t,\theta)] = \sum_{1\leqslant L\leqslant N_p}\left[\chi_{l_1\ldots l_{N_\theta}}(t)\right]^2 \prod_{p=1}^{N_\theta} a_{l_p} \tag{33}$$

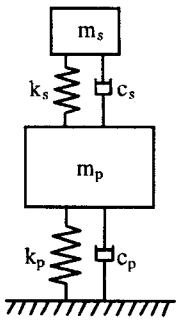
Notice that the overall mean-square response $\sigma_\rho^2(t)$ depends only on the zeroth coefficient $\chi_{0\ldots0}(t)$ given by equation (31). For stationary response, each $\chi^{(rs)}_{l_1\ldots l_{N_\theta}}$ is independent of the time t, and is given in this case by a linear algebraic system of equations of the type (26) with $\dot{\underline{z}}^{(r)} = 0$.

Although, no simple formulas exist for computing the probability of level exceedance given by equation (18), the orthogonal expansion provides an alternative for computing these quantities efficiently using simulations and avoiding repeated analyses of structural response.

NUMERICAL EXAMPLES

2-DOF system

Consider the system shown in Figure 1, consisting of two oscillators connected in series. The behavior of this simple structure is representative of the behavior of more general multi-degree-of freedom structures with a wide variety of dynamic characteristics, including primary-secondary systems. The mass ratio $\eta = m_s/m_p$ and the frequency ratio $\alpha = \omega_s/\omega_p$ are important variables controlling the system dynamics.[31] For illustration purposes, the mass ratio is taken to be $\eta = 0{\cdot}01$ so that the system is representative of a 2-DOF primary–secondary structure. The following nominal values are assumed for the system parameters: $\zeta_p = 0{\cdot}05$, $\zeta_s = 0{\cdot}02$ and $\omega_p \equiv \sqrt{k_p/m_p} = 1$. The system governing the evolution of the response has two complex conjugate modes. The linear system governing the evolution of the moments of the response has $n = 2$ real modes and $n^2 = 4$



$$\omega_s = \sqrt{\frac{k_s}{m_s}}, \qquad \zeta_s = \frac{1}{2} \cdot \frac{c_s}{\sqrt{m_s k_s}}$$

$$\omega_p = \sqrt{\frac{k_p}{m_p}}, \qquad \zeta_p = \frac{1}{2} \cdot \frac{c_p}{\sqrt{m_p k_p}}$$

Figure 1. Primary–secondary system

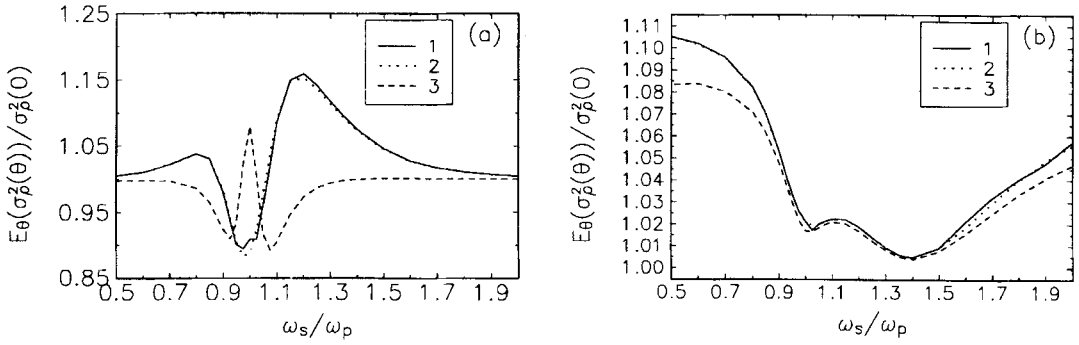


Figure 2. Stationary mean-square response ratio; (a) uncertain k_s, (b) uncertain ζ_s; (1) proposed, (2) 'exact' numerical integration, (3) perturbation

complex conjugate pairs of modes. The response quantity of interest $\rho(t)$ is chosen to be the spring force of the secondary oscillator, i.e. $\rho(t) = k_s u(t)$, where $u(t)$ is the displacement of the mass m_s relative to the mass m_p.

The uncertainty in the stiffness k_s and the damping ratio ζ_s of the secondary oscillator are separately considered in order to determine their relative effect on the response. The parameterization for k_s is chosen to be of the form $k_s = k_{s0}(1 + v_{k_s}\theta_1)$, where $\theta_1 \equiv \theta_{k_s}$ is assumed to be a uniformly distributed random variable with zero mean and unit variance, and v_{k_s} is the coefficient of variation of k_s. A similar parameterization $\zeta_s = \zeta_{s0}(1 + v_{\zeta_s}\theta_2)$, where $\theta_2 \equiv \theta_{\zeta_s}$ has same probability density function as θ_1, is considered for the case where ζ_s is uncertain. A representative case is considered in which the coefficients of variation of the uncertain parameters k_s and ζ_s are chosen to be $v_{k_s} = 0{\cdot}10$ and $v_{\zeta_s} = 0{\cdot}30$, respectively.

The accuracy of the proposed method is first illustrated for a white noise base excitation and for values of the frequency ratio $\alpha = \omega_s/\omega_p$ ranging from 0·5 to 2·0. A cubic polynomial expansion for the modal quantities and a cubic orthogonal series expansion, $N_e = 3$, for each modal response is used. Figures 2(a) and 2(b) show the ratio of the stationary mean-square response $\sigma_\rho^2(t) = E_{\underline{\theta}}[\sigma_\rho^2(t, \underline{\theta})]$ computed for the uncertain system with uncertainties in θ_{k_s} and θ_{ζ_s}, respectively, to the stationary mean-square response $\sigma_\rho^2(t, \underline{0})$ corresponding to the nominal system involving no uncertainties. For comparison purposes, results obtained from the second-order perturbation method[28] and numerical integration method are also included in this Figure. The numerical integration is referred to as "exact" since it converges to the exact solution as the number of integration points is increased. Figure 3 makes similar comparisons for the COE of the stationary response $\rho(t, \underline{\theta})$. Values of COE significantly different from zero suggest that the response process is non-Gaussian.

The reliability level b_{95} corresponding to 95 per cent probability of not being exceeded by the stationary response $\rho(t, \underline{\theta})$ over a duration of $20T_s$, where $T_s = 2\pi/\omega_s$ is the period of the secondary oscillator, is also computed using equation (18) with $L_g(b, t, \underline{\theta})$ given by equation (11). To demonstrate the importance of non-Gaussianity, the reliability levels b_g corresponding to 95 percent probability of not being exceeded are also

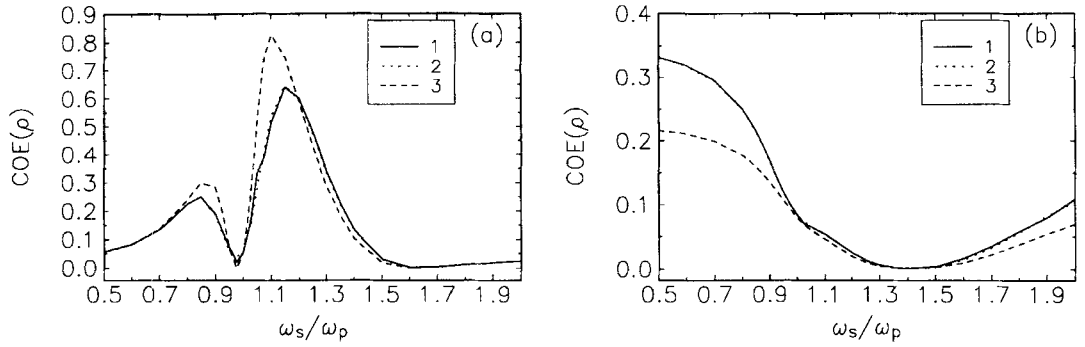


Figure 3. Stationary coefficient of excess; (a) uncertain k_s, (b) uncertain ζ_s; (1) proposed, (2) 'exact' numerical integration, (3) perturbation

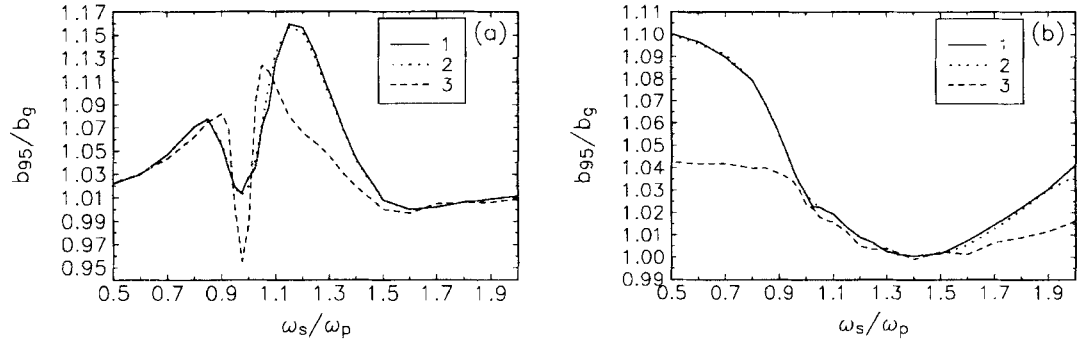


Figure 4. Reliability ratio b_{95}/b_g; (a) uncertain k_s, (b) uncertain ζ_s; (1) proposed, (2) 'exact' numerical integration, (3) perturbation

computed from equation (11) using only the unconditional second-order statistics of the response, that is, assuming that the response is Gaussian. The proposed, "exact", and perturbation methods are used to compute the variability in terms of $\underline{\theta}$ of the second-order statistics of the response, involved in the computation of b_{95} and b_g. The results obtained for the ratio b_{95}/b_g from these methods are shown in Figure 4. The deviation of the ratio b_{95}/b_g from unity indicates the importance of non-Gaussianity on the reliability.

Uncertainties would be important if the deviation of the ratios, shown in Figures 2 and 4, from unity is of the same order as the coefficients of variation describing the parameter uncertainties. Therefore, the results in Figures 2 and 4 suggest that uncertainties in stiffness are important for primary secondary systems that are close to tuning, while uncertainties in damping ratio are important for systems in detuned conditions. Moreover, considering that the coefficients of variation assumed for θ_{k_s} and θ_{ζ_s} were 0·10 and 0·30, respectively, it can be inferred from these figures that uncertainties in stiffness have a more pronounced effect than uncertainties in damping ratios.

The very good performance of the proposed method is evident by viewing Figures 2–4. In the case of uncertain stiffness, the perturbation method provides a poor fit for primary–secondary systems close to tuning, even if the level of uncertainties is as low as $v_{k_s} = 0{\cdot}10$. In particular, the perturbation method fails to predict the effect of parameter uncertainties for the cases where these uncertainties are important.

The deviation of the ratio b_{95}/b_g from unity indicates that the effects of non-Gaussianity of the response process on the reliability are important. Moreover, the fact that $b_{95}/b_g > 1$, supports the theoretical result given by equation (17) that neglecting the effects of non-Gaussianity results in non-conservative estimates of structural reliability. It is worth observing the strong correlation between the variation of the COE given in Figure 3 and the variation of the ratio b_{95}/b_g given in Figure 4. This is not surprising since the COE is a simplified descriptor of the non-Gaussianity of the response. It should be mentioned that obtaining numerical estimates of the reliability levels b_g and the COE using the proposed methodology is substantially less time consuming than performing the numerical integrations (18) for obtaining the actual reliability levels b_{95}. Thus, the strong correlation between COE and b_{95}/b_g can be used advantageously to infer about the reliability of

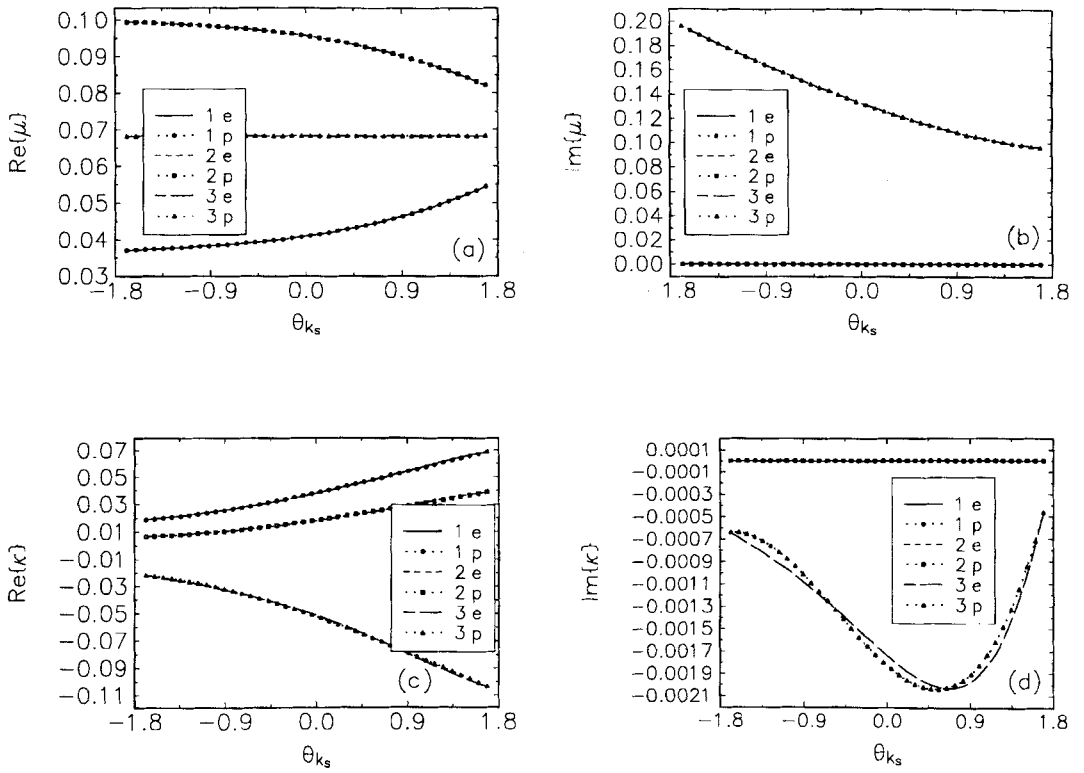


Figure 5. Variability of modal quantities involved in equation (6) due to variability of the stiffness k_5. (1) 1st mode; (2) 2nd mode; (3) 3rd mode; e =exact, p =proposed

the system by avoiding the expensive computation of b_{95}. It is suggested that first the COE be calculated and if its value is close to zero, then the reliability levels b_{95} be approximated by b_g. However, if the COE is significant, then in order to obtain confident results, one needs to proceed with calculating b_{95} from $L(b,t)$ given by the integral (18).

An insight into the factors contributing to the degree of accuracy predicted by the perturbation and the proposed method can be gained by investigating how well these methods can fit the exact variability of the response as a function of θ_{k_5} over the interval $[-\sqrt{3},\sqrt{3}]$ corresponding to non-zero probability density values for θ_{k_5}. The accuracy of the response variability for the proposed method also depends on how well the expansions for $\mu^{(r)}$ and $\kappa^{(r)}$ fit their exact variability. Comparisons of the variabilities for $\mathrm{Re}\{\mu^{(r)}\}$, $\mathrm{Im}\{\mu^{(r)}\}$, $\mathrm{Re}\{\kappa^{(r)}\}$ and $\mathrm{Im}\{\kappa^{(r)}\}$ are shown in Figure 5 for the first three lower modes involved in the modal expansion (6). The variability of the stationary response as a function of θ_{k_5} is plotted in Figure 6. The variability predicted by the proposed method is plotted for values of $N_e = 2$ and 3 and it is found to fit well the exact variability over the whole range of variation of θ_{k_5}. The perturbation method provides a poor fit over the whole range θ_{k_5} with the exception of a small interval in the neighborhood of $\theta_{k_5} = 0$.

The lack of accuracy in fitting the response variability as a function of the system uncertain parameters over the whole range of variation of these parameters is the main reason for which the perturbation method fails to provide good estimates of the response statistics. The second-order perturbation method was consistently found to provide the poorest fit. In contrast to the proposed method, increasing the order of the expansion in the perturbation method does not necessarily improve the fit.[15] It should be emphasized that secular terms, often considered as the main cause of the poor performance of the perturbation method,[17,28] are not present in the cases considered in this example since the stationary response is obtained by solving time invariant algebraic equations for the response moments. The existence of the secular terms in time varying or non-stationary problems will certainly worsen the performance of the perturbation method.

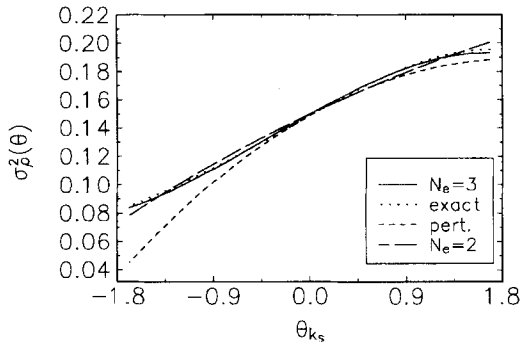


Figure 6. Variability of the variance of the stationary response due to variability of the stiffness k_s; Uniform distribution

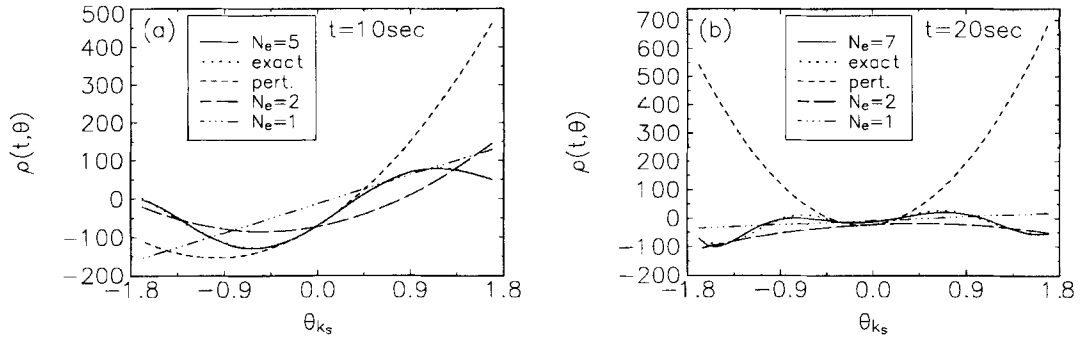


Figure 7. Variability of the response due to variability of the stiffness k_s; (a) $t = 10$ s; (b) $t = 20$ s; deterministic excitation; uniform distribution

Next, a deterministic earthquake ground motion input is considered. The North–South component of the 1940 El Centro earthquake record is taken as the base acceleration. The nominal values for the system parameters are kept the same as before. The frequency ratio is taken to be $\alpha = 0{\cdot}9$ and the random variable θ_{k_s} is assumed to be uniformly distributed with $v_{k_s} = 0{\cdot}1$. The exact variability of the response at times $t = 10$ s and $t = 20$ s is compared in Figure 7 to the approximate ones computed using the second-order perturbation method as well as using the proposed methodology with different orders N_e chosen for the polynomial expansion (22). Again, the proposed method provides the best fit, on the average, over the whole domain of variation of the uncertain parameter. In contrast, the perturbation method provides a good approximation in the neighborhood of $\theta_{k_s} = 0$ and a very poor approximation away from it. The proposed method can provide results with any desired degree of accuracy by using sufficiently large values for N_e.

The mean and standard deviation of the response are shown in Figure 8. The order of the approximation for the proposed method is chosen to be as low as $N_e = 2$ for the mean response and $N_e = 5$ for the standard deviation of the response. The very good fit to the mean response provided by the proposed method with $N_e = 2$ is due to the fact that the mean response depends only on the zeroth order term of the polynomial expansion (22). Increasing the value of N_e from 2 to higher values does not affect significantly the value of the zeroth order term. Usually accurate predictions of the mean response are obtained by choosing the value of N_e as low as $N_e = 2$ or, in some cases, even $N_e = 1$. The very poor fit obtained using the perturbation method is partly due to its inadequacy in capturing the response variability as shown in Figure 7 and partly due to the presence of the secular terms,[17,28] especially for the earliest part of the response.

Similar studies conducted for the case of uncertain damping ratio ζ_s with coefficient of variation 0·3 showed that the mean and the standard deviation of the response can be accurately modelled by choosing $N_e = 1$ and 2, respectively, although the uncertainty assumed for ζ_s is three times larger than that assumed for k_s. This is because the variation of the response as a function of the damping ratio is much smoother than its variation as a function of the stiffness.

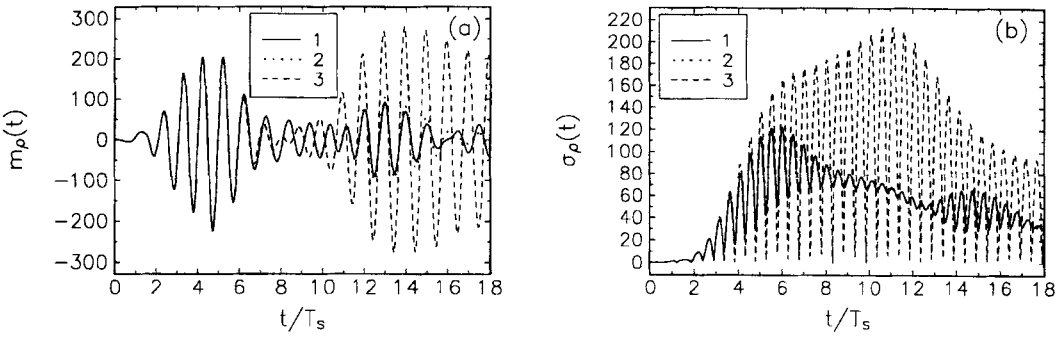


Figure 8. Comparison between (1) proposed method, (2) 'exact' numerical integration method, and (3) perturbation method: (a) mean ($N_e = 2$), (b) standard deviation ($N_e = 5$); uniform distribution; $N_\theta = 1$

In general, the proposed method can provide results with any desired degree of accuracy by using sufficiently large values for N_e in the series expansion (22). The value of N_e was found to be dependent on the frequency of the mode considered in equation (4) or (6). The lower the frequency of a mode, the lower was found to be the required value of $N_e^{(r)}$. This can be advantageously used to speed up computations by considering different values $N_e^{(r)}$ for different modes included in the analysis.

Ten-story building

The second system is a ten-story building modeled as a 10-DOF chain-like system of masses connected by springs. The building is excited at the base. All floor masses are taken as known and equal to $m_i = m = 1$. The interstory stiffnesses k_i, $i = 1,\ldots,10$ are assumed to be uncertain. The uncertain vector $\underline{k} = [k_1,\ldots,k_{10}]^T$ is modelled as a Gaussian vector of dependent variables with mean equal to $[1,\ldots,1]^T k_0$, where $k_0 = 10\,000m\,\mathrm{s}^{-2}$, and covariance matrix $\mathbf{H}$ given by its components $H_{ij} = E[(k_i - \bar{k}_i)(k_j - \bar{k}_j)] = k_0^2 c^2 \exp[-(i-j)^2/\lambda^2]$, where c is a non-dimensional measure of the level of uncertainty, analogous to a coefficient of variation, and $\lambda = 1/4$ is a non-dimensional measure of the correlation length of the random vector $\underline{k}$. Spectral decomposition[23] based on the covariance matrix corresponding to the random vector $\underline{k}$ is employed to describe the uncertainty in $\underline{k}$ by a set of ten independent Gaussian variables. A spectral truncation method, described in References 18 and 23, is further used to reduce to five the number of important variables needed to accurately describe the random vector $\underline{k}$. Rayleigh damping with $\mathbf{C} = \alpha\mathbf{M} + \beta\mathbf{K}$ is assumed, where the values of α and β are determined by assigning the values for the damping coefficients of the first two lower modes. These first two damping coefficients are assumed to be uncertain, fully correlated, and their uncertainty is modeled by a Gaussian distributed random variable with mean value 0·05 and coefficient of variation 20 per cent. Notice that in this case $\mathbf{C}(\underline{\theta})$ is a non-linear function of $\underline{\theta}$. The total number of uncertain parameters of this system is $N_\theta = 6$. The response quantity of interest is taken to be the displacement of the top floor.

The mean and the standard deviation of the response due to the deterministic 1940 El Centro earthquake excitation, computed for level of uncertainty $c = 0{\cdot}10$, are shown in Figures 9(a) and 9(b), respectively. Only the first two modes were found to be significantly contributing to the response. A quadratic approximation was used to model the variability in the modal quantities of the first two modes. The truncation order $N_e^{(r)}$ in the expansion (22) was chosen to be $N_e^{(r)} = 5$. The results from the simulation method obtained using 20 000 samples and the second-order perturbation method are also included in these figures for comparison. The perturbation method gives poor results for the standard deviation of the response. The results obtained by the proposed method fit well those obtained using simulations.

The case of white noise stochastic excitation applied at the base is also considered to illustrate some of the computational advantages of the modal based methodology proposed herein. The system governing the covariance response for a given value of $\underline{\theta}$ has dimension $n(n+1)/2 = 210$ since in this case $n = 20$. The total number of modes involved in equation (6) is $n + 2n^2$, where $n = 10$ are real and the remaining $2n^2 = 200$

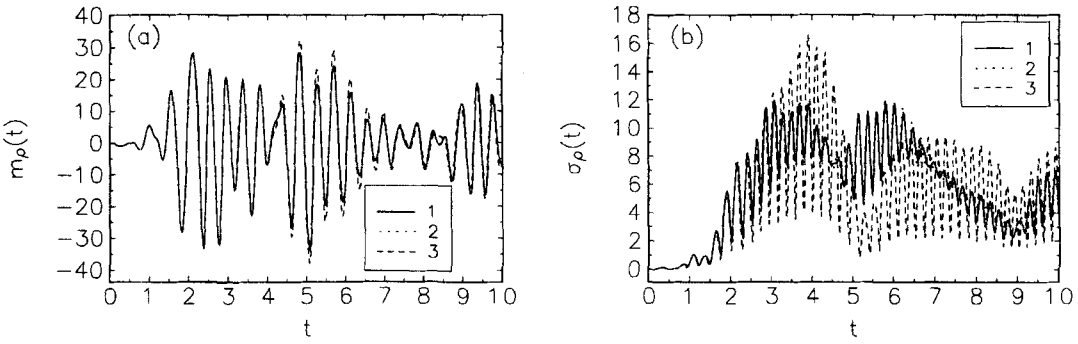


Figure 9. Comparison between (1) proposed method ($N_e = 5$), (2) simulations, and (3) perturbation method: (a) mean, (b) standard deviation; Deterministic excitation; Gaussian distribution; $N_\theta = 6$

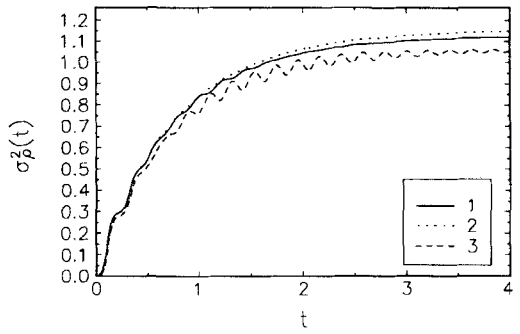


Figure 10. Response variance; (1) proposed method ($N_e = 1$), (2) simulations, and (3) perturbation method; Stochastic excitation; Gaussian distribution; $N_\theta = 6$

appear in complex conjugate pairs. However, only the contributing terms in the modal expansion (6) need to be included in the analysis of the uncertain system. The method proposed in Reference 20 for deterministic systems subjected to stochastic excitation is used to identify the number of contributing terms. Specifically, an eigenvalue analysis is first performed to compute the modal properties of the nominal structure corresponding to $\underline{\theta} = \underline{0}$. The modes corresponding to small enough values of the modal ratio $\kappa^{(rs)}/\mu^{(rs)}$ are neglected in equation (6). Specifically, a mode (rs) is neglected if $\kappa^{(rs)}/\mu^{(rs)} < 0{\cdot}005 \max_{(rs)}\{\kappa^{(rs)}/\mu^{(rs)}\}$. Only 2 real and 4 complex conjugate pairs of modes out of the total 110 modes were found to be contributing.

Using only these six modes, the unconditional variance $\sigma_\rho^2(t)$ of the response is computed assuming a higher level of uncertainty $c = 0{\cdot}20$. Up to first-order polynomials were considered in the series expansion (22), i.e. $N_e^{(rs)} = 1$. The results for $\sigma_\rho^2(t)$ normalized by the stationary response $\sigma_\rho^2(t, \underline{0})$, corresponding to $\underline{\theta} = \underline{0}$, are shown in Figure 10 and compare well with those obtained using 200 simulations. The second-order perturbation method provides less accurate results than those provided by the proposed methodology based on a first-order polynomial expansion. Also, the spurious oscillatory behavior obtained by the perturbation method is due to secular terms.[28]

Certain advantages of the proposed method in terms of both computer time and memory can be revealed by comparing it with other OEM methods based on expansion of the state vector of the response. Using equation (23), the number of the unknown coefficients in the expansion (22) is $N_d = 7$. The proposed method requires the solution of six uncoupled systems of dimension $N_d = 7$ for each real mode and $2N_d = 14$ for each pair of complex conjugate modes. In contrast, the OEM method proposed in Reference 1 requires the solution of one system of $N_d n(n+1)/2 = 1470$ coupled differential equations of the type (26). In addition, the latter method requires that the damping matrix $\mathbf{C}$ be a linear function of the uncertain parameters which is clearly not the case in this example.

CONCLUSIONS

A modal-based probabilistic methodology was developed for calculating the response variability and statistics of multi-degree-of-freedom linear systems with uncertain parameters. The methodology is applicable to non-classically damped systems subjected to both deterministic and stochastic loads modelled by Gaussian processes. In the particular case of stochastic excitation, the statistics of the response are obtained by analyzing the variability of the second-order moments of the response in terms of the variability of the system parameters. This variability is obtained by analyzing the modes of the system governing the evolution of the covariance response.

The proposed method differs from other approaches[1,6,9,18] based on series expansions in that it analyses separately each one of the contributing modes, resulting to considerable reduction of the size of the systems governing the evolution of the coefficients present in the series expansions. Reductions in computational time and computer space are even more substantial for structures subjected to stochastic excitations since only a small number of the modes of the large system governing the evolution of the covariance response are usually contributing. The modal formulation presented is also well-suited to efficiently handle the important cases in which the mass, damping and stiffness matrices depend nonlinearly on the system uncertain parameters, and also to treat a variety of probability distributions for the system uncertainties.

It has been demonstrated that the second-order perturbation method, shown in other studies to perform well for static load cases,[10,14,23] provides poor results for systems subjected to deterministic dynamic loads. Moreover, the perturbation method, although it has been found to perform well when calculating the stationary response of systems subjected to stochastic loads and when the levels of uncertainty are relatively small,[17,28] it may lead to major inaccuracies when applied to special structural systems, such as primary–secondary systems, even when the uncertainties considered are small. The proposed methodology is consistently more accurate than the perturbation method. Such accuracy can in several cases be achieved by using a series expansion of first-order polynomials. The present method can be used to study the effect of modelling uncertainties on the response for a wide range of complex engineering systems.

ACKNOWLEDGEMENTS

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APPENDIX

The coefficients in equation (25) are given by:

$$d_i(\underline{l}) = C_1(\ell_i, l_i)\frac{a_{\ell_i}}{a_{l_i}} \tag{34}$$

$$d_{ij}(\underline{l}) = C_1(\ell_i, l_i)C_1(\ell_j, l_j)\frac{a_{\ell_i}a_{\ell_j}}{a_{l_i}a_{l_j}}, \quad i \neq j \tag{35}$$

$$d_{ii}(\underline{l}) = C_2(\ell_i, l_i)\frac{a_{\ell_i}}{a_{l_i}} \tag{36}$$

$$h_0(\underline{l}) = \prod_{p=1}^{N_0} \delta_{l_p 0} \tag{37}$$

$$h_i(\underline{l}) = \sum_{\ell_i = l_i - 1}^{l_i+1} C_1(\ell_i, l_i)\delta_{\ell_i 0} \prod_{\substack{p=1 \\ l_i \neq \ell_i}}^{N_0} \delta_{l_p 0} \tag{38}$$

$$h_{ij}(\underline{l}) = \sum_{\ell_i=l_i-1}^{l_i+1} \sum_{\ell_j=l_j-1}^{l_j+1} C_1(\ell_i, l_i) C_1(\ell_j, l_j) \delta_{\ell_i 0} \delta_{\ell_j 0} \prod_{\substack{p=1 \\ l_i \neq \ell_i, \ell_j}}^{N_0} \delta_{l_p 0}, \quad i \neq j \tag{39}$$

$$h_{ii}(\underline{l}) = \sum_{\ell_i=l_i-2}^{l_i+2} C_2(\ell_i, l_i) \delta_{\ell_i 0} \prod_{\substack{p=1 \\ l_i \neq \ell_i}}^{N_0} \delta_{l_p 0} \tag{40}$$

and $\bar{h} = \prod_{p=1}^{N_0} (a_0/a_{l_p})$, where $\underline{l} = (l_1 \ldots l_{N_0})$ identifies the equation considered in (25). The values of $C_1(i,j)$ and $C_2(i,j)$ depend on the polynomial basis chosen in the expansion (22).

REFERENCES

1. H. Jensen and W. D. Iwan, 'Response of systems with uncertain parameters to stochastic excitation', *J. eng. mech. ASCE* **118**, 1012–1025 (1992).
2. T. Igusa and A. Der Kiureghian, 'Response of uncertains systems to stochastic excitation', *J. eng. mech. ASCE* **114**, 812–832 (1988).
3. C. G. Wall and C. G. Bucher, 'Exceedance rate including system uncertainties', in G. I. Schueller (ed.), *Structural dynamics: Recent Advances*, Springer, New York, 1991, pp. 285–300.
4. Jr. B. F. Spencer and I. Elishakoff, 'Reliability of uncertain linear and nonlinear systems', *J. eng. mech. ASCE* **114**, 135–149 (1988).
5. L. S. Katafygiotis and J. L. Beck, 'A very efficient moment calculation method for uncertain linear dynamic systems', *Probabilistic eng. mech.* **10**, 117–128 (1995).
6. H. Jensen and W. D. Iwan, 'Response variability in structural dynamics', *Earthquake eng. struct. dyn.* **20**, 949–959 (1991).
7. G. Deodatis, 'Weighted integral method. I: stochastic stiffness matrix', *J. eng. mech. ASCE* **117**, 1851–1864 (1991).
8. A. Der-Kiureghian and J. B. Ken, 'The stochastic finite element method in structural reliability', *Probabilistic eng. mech.* **3**(2), 83–91 (1988).
9. R. Ghanem and P. Spanos, 'Polynomial chaos in stochastic finite element', *J. appl. mech., ASME* **57**, 197–202 (1990).
10. W. K. Liu, A. Mani and T. Belytschko, 'Finite elements methods in probabilistic mechanics', *Probabilistic eng. mech.* **2**(4), 201–213 (1987).
11. T. Takada, 'Weighted integral method in stochastic finite element analysis', *Probabilistic eng. mech.* **5**(3), 146–156 (1990).
12. E. Vanmarcke and M. Grigoriu, 'Stochastic finite element analysis of simple beams', *J. eng. mech. ASCE* **109**, 1203–1214 (1983).
13. C. G. Bucher and C. E. Brenner, 'Stochastic response of uncertain systems', *Archive appl. mech.* **62**, 507–516 (1992).
14. H. Contreras, 'The stochastic finite element method', *Comput. struct.* **12**, 341–348 (1980).
15. L. S. Katafygiotis, 'Treatment of model uncertainties in structural dynamics', *EERL Report No. 91-01*, Division of Engineering and Applied Science, California Institute of Technology, CA, 1991.
16. L. S. Katafygiotis and J. L. Beck, 'Treating model uncertainties in structural dynamics', *Proc. 9th world conf. on earthquake eng.*, Tokyo, **V**, 301–306, (1988).
17. H. U. Koyluoglou, S. R. K. Nielsen and A. S. Cakmak, 'Linear dynamic analysis of SDOF systems with random parameters subject to earthquake excitations', in A. S. Cakmak and C. A. Brebbia (eds), *SDEE'93, 6th int. conf. soil dyn. and earthquake eng.*, pp. 775–795, Bath, UK, 1993.
18. W. D. Iwan and H. Jensen, 'On the dynamic response of continuous systems including model uncertainties', *J. appl. mech.* ASME **60**, 484–490 (1993).
19. T. T. Soong and M. Grigoriu, *Random Vibration of Mechanical and Structural Systems*, Prentice Hall, New Jersey, 1993.
20. C. Papadimitriou, 'Stochastic response cumulants of MDOF linear systems', J. eng. mech. *ASCE* **121**, 1181–1192 (1995).
21. S. O. Rice, 'Mathematical analysis of random noise', *Bell system tech. J.* **23**, 282–332 (1944); **24**, 46–156 (1945). Reprinted in *Selected Papers on Noise and Stochastic Processes*, N. Wax (ed.), Dover, New York (1954).
22. P. D. Spanos and R. Ghanem, 'Stochastic finite element expansion for random media', *J. eng. mech. ASCE* **115**, 1035–1053 (1989).
23. W. K. Liu, T. Belytschko and A. Mani, 'Random fields finite elements', *Int. j. numer. methods eng.* **23**, 1831–1845 (1986).
24. R. W. Clough and J. Penzien, *Dynamics of Structures*, 2nd edn, McGraw-Hill, New York, 1993.
25. S. R. Winterstein, 'Nonnormal responses and fatigue damage', *J. eng. mech. ASCE* **111**, 1291–1295 (1985).
26. J. F. Baldwin and S. G. Hutton, 'Natural modes of modified structures', *AIAA j.* **23**, 1737–1743 (1985).
27. J. D. Collins and W. T. Thompson, 'The eigenvalue problem for structural systems with uncertain parameters', *AIAA j.* **7**, 642–648 (1969).
28. H. U. Koyluoglu, 'Stochastic response and reliability analyses of structures with random properties subject to stationary random excitation', *Report*, Department of Civil Engineering and Operations Research, Princeton University, NJ, 1995.
29. G. A. Korn, *Mathematical Handbook for Scientists and Engineers*, 2nd edn, McGraw-Hill, New York, 1968.
30. M. Rosenblatt, 'Remarks on a multivariate transformation', *Ann. math. statist.* **23**, 470–472 (1952).
31. T. Igusa and A. Der Kiureghian, 'Dynamic response of multiply supported secondary systems', *J. eng. mech. ASCE* **111**, 20–41 (1985).